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NUMBER AND FRACTIONS.

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A clear understanding of what *number* is and what gives rise to the number idea removes all difficulty from the grasping of the *fraction* idea.

Number does not inhere in objects, cannot be perceived by the senses; otherwise the mere presentation of 2, 3......n objects to the senses would give rise to the idea of number. There is in every sound mind a *measuring* instinct, which, in the nature of things, is just as essential to life and progress as is memory. Both the physical and ideal worlds are full of entities—vague wholes—which the mind must *measure* for the purpose of making them more definite. Measuring requires a "unit of measure." Naturally the first measurements made by a child are vague; as when he measures (counts) the chairs in a room, the marbles in his pocket, the fingers on his hand. His units of measure—chair, marble, finger—are indefinite, as are the results of his processes. A later stage involves *exact* measurements; i. e., an exactly defined unit of measure is used. A whole (of quantity), say a piece of cloth, is to be measured—made definite in value. A *yard* (exactly defined as 3 feet or 36 inches) is taken as the unit and applied (say) *ten* times. Then *ten* repetitions of the unit is the *number*. Considered by itself the *ten* is *pure number*, the result of a purely mental process; it expresses the *ratio* of the measured *quantity* to the measuring unit. Applied to the unit of measure, then ten expresses the numerical value of the measured quantity—10 yards of cloth. This ten yards, it is evident, is *quantity*, not number. It is what arithmetics erroneously call "concrete number." In this example the pure number indicates either of two things: (a) that the unit is taken *ten times*, or (b) that ten parts (units) are taken *one time*. It answers the question "how many?" Applied to the unit, it answers the question "how much?"

The number and unit of measure *together* give the absolute magnitude of the quantity; the number *alone* gives the relative value. Hence we may say that *number is the ratio of the quantity measured to the unit of measure.*

It is plain that *any* quantity may be used as a unit of measure. Measurement is more exact when this unit is itself made up of a definite number of equal parts—measured by some other unit, which may be called "primary" to distinguish it from the actual or direct unit of measure, which may be called "derived." Thus, if the unit of measure is three feet and it is taken ten times, we have the primary unit *one foot*, the derived unit *three feet*, and the number of derived units, *ten*. We have ten *threes*. To find the number of primary units we use multiplication, which gives thirty *ones*; the quantity is now more definite.

Again, in the quantity $5\times\$3$, the primary unit is \$1, the derived (direct, actual) unit \$3, five of which=15 primary units.

The derived unit is not necessarily a *multiple* of the primary unit; it may be one or more of its *equal parts*. Thus in $\$\frac{5}{2}$, the primary unit is, as above, \$1, while the derived unit is $\$\frac{1}{2}$, the number of them *five*. The fraction $\frac{5}{2}$ expresses the ratio of the measured quantity ($\$\frac{5}{2}$) to the primary unit (\$1). The numerator shows how many derived units make up the quantity, the denominator shows the relation between the derived and primary units. It is thus seen that the fraction involves no new idea. Its notation is more complete than that of the integer in that it defines the derived unit—makes *explicit* what is implied in the integral notation. This appears in the processes of finding the value of 5 hats (*a*) at \$3 each, (*b*) at $\$\frac{1}{2}$ each.

$$5\times\$3=5\times\overline{3\times\$1}=15\times\$1=\$15.$$

$$5\times\$\tfrac{1}{2}=5\times\overline{\tfrac{1}{2}\times\$1}=\tfrac{5}{2}\times\$1=\$\tfrac{5}{2}.$$

The denominator 2 shows the relation between the derived unit ($\$\frac{1}{2}$) and the primary unit (\$1). In \$15, however, there is nothing to show the relation between \$3 and \$1. (This is seen in $5\times\overline{3\times\$1}$). In no other respect does the fraction differ from the integer. Both 15 and $\frac{5}{2}$ express ratio to the primary unit \$1. The 15 shows the number of primary units, but not that of the derived units. The $\frac{5}{2}$ shows both; there are 5 derived units, $\frac{5}{2}$ primary units.

In view of these facts it appears that a correct definition of *number* includes that of *fraction*, which is simply a number whose notation gives a more complete statement of the mental processes by which number is constituted. For mathematical purposes *Newton's* definition cannot be much improved: "Number is the abstract ratio of one quantity to another quantity of the same kind." Ratio being a pure abstraction, the word "abstract" should be omitted. Euler says, "Number is the ratio of one quantity to another quantity taken as unit." Drs. McLellan and Dewey define number as, "The repetition of a certain magnitude used as the unit of measurement to equal or express the comparative value of a magnitude of the same kind."*

*In conclusion I wish to say that every live teacher should read "*The Psychology of Number*,"

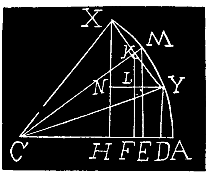
It is clear that $\frac{1}{n}$ of any magnitude may be repeated as a unit just as well as $\frac{n}{n}$ or $\frac{3n}{n}$; it is equally plain that $\frac{m}{n}$ is as much an expression of ratio as is m. Hence each definition applies to fractions as well as integers.

It is neither necessary nor advisable to divide ("break") single things (individuals, as apples) into parts in order to get fractions. In counting the eggs in a dozen (e. g.) the wee bairn is on the border of the fairyland of fractions, though he may not be conscious of it. At any stage of his counting the result is either integral or fractional. Five eggs is integral with respect to the *unit* (1 egg); it is fractional with respect to the *unity* or whole (dozen)—5 out of 12, 5 *twelfths*. Five *half-yards* is just as integral as 5 *yards*. The *ratio* in each is *five*. But in $\frac{5}{2}$ yards the ratio is $\frac{5}{2}$; the fractional idea is present, owing to the denominator, which defines the unit of measure.

SOME TRIGONOMETRIC RELATIONS PROVED GEOMETRICALLY.

By P. H. PHILBRICK, C. E., Pineville, Louisiana.

Most trigonometric formulæ may be proven geometrically in an elegant manner; and moreover, the relations between the trigonometric functions may be shown at a glance by means of the geometric figures. The results are all the more interesting, too, when proven also directly from first principles. For this reason the following exercises are offered.



For convenience, describe the arc AYX, and take the radius AC for the unit of measurement. Let the arc $AX=x$ and arc $AY=y$. Take M at the middle of XY, and draw lines as indicated.

Then $DY=\sin y$, $HX=\sin x$, $EM=\sin\frac{1}{2}(x+y)$, $KY=\sin\frac{1}{2}(x-y)$, $NX=\sin x-\sin y$, $NY=\cos y-\cos x$, $CE=\cos\frac{1}{2}(x+y)$, $CK=\cos\frac{1}{2}(x-y)$.

$$\text{Now, } HX+DY=2KF=2EM\frac{KF}{EM}=2EM\frac{CK}{CM}=2EM.CK.$$

That is, $\sin x+\sin y=2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$............(1).

by Drs. McClellan and Dewey. It is interesting in matter, vigorous and aggressive in style, refreshing in its originality, and scholarly in its conception and execution. It is in the 33d volume of the International Education Series, published by D. Appleton & Co., New York.